A State-space and Constrained GEKS Approach to the Construction of Panels of Real Incomes at Current and Constant Prices

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Abstract

The paper provides an overview of the status of our research on the development of an econometric approach to the construction of panels of PPPs for the purpose of spatio-temporal comparisons of prices and real incomes. We present an overview of the approach to construct a panel of PPPs at current prices and present empirical estimates obtained using 180 countries and the period 1970-2010 from the forthcoming UQICD Mark II. A GEKS based method satisfying fixity of PPPs of currencies in a given year is proposed and a closed form derived from which a full panel of space-time consistent PPPs can be computed.

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1 Introduction

The main objective of the paper is to briefly describe the progress in the development of an econometric framework for the extrapolation of purchasing power parities (PPPs) to construct panels of PPPs that can be used for cross-country comparisons of real incomes in any given year (at current prices) and across space and time where real incomes are expressed relative to a reference country and benchmark year. Our focus has been on the development of a method that can combine: information on PPPs generated by various benchmarks of the International Comparison Program (ICP); published data from national sources on movements in prices at the country-level in the form of deflators for the gross domestic product (GDP); and the past efforts in the modelling for extrapolating PPPs to

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countries that have not participated in the ICP benchmarks through the use of models of national price levels. The pioneering work of Summers and Heston [1991], Heston et al. [2006] and Heston et al. [2009] has led to the widely used Penn World Tables (PWT) which are compiled using some of these elements. In particular, the extrapolations rely to a large degree on the latest benchmark information available and, therefore, the constructed panels could be influenced by specific benchmarks. The work on the new generation PWT8.0 to be reported in the paper by Feenstra et al. [2013] is moving in the direction of using PPPs from different benchmarks in the extrapolation process.

Inspired by the enormous contribution made by Heston, Summers and Aten through the development of PWT, we have worked on the development of a more formal structure for the generation of panels of PPPs and panels of real incomes at current and constant prices. As an initial step, we formulated an econometric approach leading to a model that can be expressed in a state-space framework (Rao et al., 2010b, Rambaldi et al., 2010, Rao et al., 2010a). The framework allows for the generation of optimal predictors (in a mean square error sense) of PPPs that make use of information available from a number of different sources (listed above). So, far the main focus has been the development of the method (model and estimation) and the study of various analytical properties of the method. The first phase of our work on the generation of a panel of PPPs at current prices has been completed. The second phase of the project had the objective of constructing time-space consistent panels of PPPs at constant prices, and we report on the progress of this work in this paper. The third phase is to extend the extrapolation methodology to the three main components of domestic absorption, viz., consumption; investment and government, which is underway.

The paper presents results on progress made thus far. Section 2 establishes basic concepts and terminology necessary for the presentation of the rest of the paper. Section 3 provides a brief overview of the work completed on the construction of consistent panels of PPPs and real incomes at current prices. The econometric approach used in the construction is discussed along with its analytical properties and numerical results for some selected countries are also presented. The project's established website, UQICD, `https://uqicd.economics.uq.edu.au/`[1] and results from UQICD 2.0 are presented. Section 4 is devoted to the problem of construction of panels of real incomes at constant prices[2], using a constrained GEKS approach. The paper is concluded with a few remarks in the last section.

2 Basic concepts and terminology

As the work focuses on the aggregate, gross domestic product (GDP), we let GDP_{it} represent GDP in country i in period t expressed in local or national currency units. These GDP aggregate measures are not comparable across

[1]The establishment of the website fulfills one of the main funding requirements from the Australian Research Council.

[2]This problem also translates into the construction of panels of PPPs over time and space with some basic consistency requirements.

countries or over time as they are influenced by price levels in the respective countries and time periods.

Let XR_{it} and PPP_{it} respectively denote the exchange rate and the purchasing power parity of the currency of country i which is equivalent to one unit of currency of a reference or numeraire country[3]. The *nominal and real* GDP of country i in period t, respectively, denoted as NGDP and RGDP expressed in the currency units of a reference country[4]

$$NGDP_{it} = \frac{GDP_{it}}{XR_{it}} \tag{1}$$

and

$$RGDP_{it} = \frac{GDP_{it}}{PPP_{it}} \tag{2}$$

The *NGDP* adjusts for differences in currency units. In contrast, RGDP adjusts for differences in currency units as well as purchasing powers of currencies based on differences in price levels observed in different countries[5]. We note a few features of the real GDP series.

1. $RGDP_{it}$ is comparable and additive across countries *at a given period* t but not for countries at different points of time. It is possible to compute regional totals for the period t.

2. $RGDP_{it}$ is not comparable to $RGDP_{ks}$ for and any i and k. Thus $RGDP_{it}$ may be termed real GDP series at *current (period t) prices*. However, this does not necessarily mean that there is a set of prices which can be used as reference prices in deriving the real GDP series.[6]

3. $RGDP_{it}$ and PPP_{it} are typical outputs of the ICP for a given benchmark year.

4. $RGDP_{it}$ is obtained by deflating the GDP by a suitable spatial price deflator, here it is PPP_{it}.

In this paper we refer to PPP_{it} and $RGDP_{it}$ series for periods $t = 1, 2, 3, \ldots, T$ and $i = 1, \ldots, N$ as a panel of PPPs and real incomes at *current or period t prices* to emphasize the fact that these PPPs and real GDP aggregates are not comparable over time. The problem of construction of these series at current prices has been satisfactorily addressed by the PWT or by the econometric approach proposed in Rao et al. [2010b,a], and for the purpose of this paper we will denote by $P\hat{P}P_{it}$ the predictions of PPP_{it} constructed from either of these approaches[7].

Now let $PPP_{it}^{k\tau}$ represent the PPP for the currency of country i in period t with reference country k and reference period τ . Then, the real GDP expressed at constant τ year prices with reference country k is given by:

[3]We drop the subscript for the reference country to keep the notation simple.
[4]A superscript to indicate the reference country would be useful but suppressed to facilitate notational brevity.
[5]RGDP is basically a volume measure, a concept advocated for use by the System of National Accounts (SNA) and emphasised in McCarthy [2013]
[6]See Rao and Balk [2013] for a definition of real income comparisons at a set of reference prices and for examples where deflated series could be interpreted as real income comparisons at some reference prices. For example, the GK based real GDP figures could be considered as real income comparisons obtained at GK international prices along with a Leontief utility function and real series obtained by using the Tornqvist index as the deflator corresponding to real income comparisons based on translog cost function.
[7]Any panel of PPPs at current prices can be used as a starting point.

$$CRGDP_{it}^{k\tau} = \frac{GDP_{it}}{PPP_{it}^{k\tau}} \quad (3)$$

Here GDP of country i in period t is adjusted for price movements over time (from the reference or base year, τ) and across space to adjust for price level differences between country i and the reference country, k. CRGDP by construction can be summed over countries as well as time periods[8].

Given these definitions and the underlying notation, the main problem is one of constructing panels of PPPs and real incomes at *constant* reference country k period t prices.

3 An Econometric Approach to the Construction of Panels of PPPs and Real Incomes at Current Prices

This section draws heavily from the descriptions of our method fully described in Rao, Rambaldi and Doran (RRD) (Rao et al. [2010a,b]). In order to avoid duplication of the material contained in these papers, only a brief description of the method is provided here. The RRD approach is designed to combine PPP data available from all the benchmarks of the International Comparison Program (ICP), since 1970 to the extensive coverage of countries in the 2005 round of the ICP, with the information on deflators at the aggregate GDP level available from the national accounts data published by the countries. In addition to these two main sources, the approach also makes use of the vast literature on the explanation of national price levels [9] in the form of a regression model which is used in extrapolating PPPs for countries that have not participated in each of the benchmarks of the ICP.

3.1 The Model

The econometric problem is one of signal extraction. That is, there are a number of sources of "noisy" information that can be combined to extract the signal. A state-space (SS) is a suitable representation for this type of problems. At any time period t the N countries can be placed in one of three groups (where t is an ICP benchmark year) or in two groups otherwise. In an ICP year, the groups are: the reference country (without loss of generality this is set to the first country), the non-participating countries and the participating countries. In a non-ICP benchmark year there are only two groups: the reference country and all others.

The mapping is from what is observed or measured *with some error* at time t to a vector of true but unobserved PPPs to be estimated. It is convenient to work with log transformations and thus, at each t the vector of log PPPs, at current prices, (for the N countries) is denoted by $p_t = ln(PPP_t)$, with elements $p_{it} = ln(PPP_{it})$ for

[8] These series are similar to the GDP series at constant prices produced by national statistical offices except that the focus in such cases is on a single country.

[9] National price level is defined as the ratio of PPP to the exchange rate of the currency of a given country.

$i = 1, ..., N$. The objective is to estimate p_t for all N countries and $t = 1, ..., T$ time periods to generate a complete panel. The mapping equations (known as a observation and transition equations in the state-space literature) are given in equations (4) and (16). The rest of the sub-section presents the economic and econometric framework that leads to these two sets of equations. Equation (4) simply links the *observed* information and noise to the latent p_t. Equation (16) provides the law of motion of p_t over time, which is derived from index theory and is the established updating approach used by PWT and Maddison [2007].

$$y_t = Z_t p_t + \zeta_t \quad (4)$$

In an ICP benchmark year the mapping is as follows,

$$y_t = \begin{bmatrix} 0 \\ \hat{p}_t \\ \tilde{p}_t \end{bmatrix} ; \; Z_t = \begin{bmatrix} S_1 \\ S_{np} \\ S_p \end{bmatrix} ; \; \zeta_t = \begin{bmatrix} 0 \\ S_{np} v_t \\ S_p \xi_t \end{bmatrix} \quad (5)$$

where,

y_t is a vector of *observed* information

$\tilde{p}_t$ is a vector of log transformations of the ICP PPP benchmarks for participating countries, $\tilde{PPP}_t$.

$\hat{p}_t$ is a vector of log PPP regression predictions for non-participating countries. The predictions are based on a model of the log of price levels ($\ln(PPP_{it}/XR_{it})$), some details provided below;

The first element of y_t is zero as that is the observation for the reference country $p_{1t} = ln(PPP_{1t})$ which is a constraint in the system

Z_t is a partitioned selection matrix with components which select the reference country (country 1), S_1, the non-participating countries, S_{np}, and the participating ICP countries, S_p; and

ζ_t is a random vector capturing the uncertainty arising from each set of sources of observed values of PPP_{it}. The first row is zero as it represents the reference country constraint. The non-participating countries have error v_t, and the ICP measures have error ξ_t. The variance-covariance matrix of ζ_t is then given by,

$$E(\zeta_t \zeta_t') \equiv H_t = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sigma_u^2 S_{np} \Omega_t S_{np}' & 0 \\ 0 & 0 & \sigma_\xi^2 S_p V_t S_p' \end{bmatrix} \quad (6)$$

In a non-benchmark years there are no observations from the ICP, thus the only observations are those produced by the predictions from the price level model and the constraint,

$$y_t = \begin{bmatrix} 0 \\ S_{np}\hat{p}_t \end{bmatrix}; \ Z_t = \begin{bmatrix} S_1 \\ S_{np} \end{bmatrix}; \ \zeta_t = \begin{bmatrix} 0 \\ S_{np}v_t \end{bmatrix} \tag{7}$$

The components of this mapping are derived from the following theoretical considerations,

1. The observed PPPs from the ICP, in the benchmark years, are related to the true PPPs through the following equation:

   $$\tilde{p}_{it} = p_{it} + \xi_{it} \tag{8}$$

   where ξ_{it} is a random error accounting for measurement error with the properties:

   $$E(\xi_{it}) = 0; \ E(\xi_{it}^2) = \sigma_\xi^2 V_{it} \tag{9}$$

   The measurement error variance-covariance is of the form

   $$V_t = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_{1t}^2 jj' + diag(\sigma_{2t}^2, \ldots, \sigma_{Nt}^2) \end{bmatrix}$$

   where j is a vector of 1's and σ_{it}^2 is the variance of the PPP from the ICP benchmark for country i in period t. Here σ_{1t}^2 is the variance of the reference country (country 1). In the empirical implementation of the method, σ_{it}^2 is assumed to be inversely related to the GDP of country i in period t [10].

2. The numerical value of the PPP for the reference/numeraire country, 1, is set at 1. Thus

   $$p_{1,t} = 0; \ t = 1, 2, \ldots, T \tag{10}$$

3. The key element of the approach is the regression model used in extrapolating PPPs to non-participating countries using PPP data from the ICP benchmarks. The regression model draws on the literature on the explanation of national price levels (Kravis and Lipsey [1983]; Clague [1988] and Bergstrand [1991, 1996]). A linear model in logarithms of price levels is postulated as below:

   $$r_{it} = ln(PPP_{it}/XR_{it}) = \beta_{0t} + \mathbf{x}'_{it}\beta_s + u_{it} \tag{11}$$

   for all $i = 1, 2, \ldots, N$ and $t = 1, 2, \ldots, T$

[10] In order to avoid circularity, NGDP (see equation (1) is used in the estimation process.

Deviating from the usual assumptions on the disturbance term, we assume that errors in (11) are spatially autocorrelated . The following specification is used

$$u_t = \phi W_t u_t + e_t \tag{12}$$

where $|\phi| < 1$ and $W_t(N \times N)$ is a spatial weights matrix. The term spatial in the present contexts refers to socio-economic distance rather than the traditional geographical distance. It follows that $E(u_t u_t')$ is proportional to $\Omega_t = (I - \phi W_t)^{-1}(I - \phi W_t)^{-1'}$. If estimates of parameters in (11) are available, then predictions of PPPs consistent with price level theory can be generated for any country in any period. These are given by:

$$\hat{p}_{it} = \hat{\beta}_{0t} + \mathbf{x'}_{it}\hat{\beta}_s + ln(XR_{it}) + \hat{\phi} W_t \hat{u}_t \tag{13}$$

Inspection of equation (11) shows it is possible to obtain estimates of the parameters by using the unbalanced panel available through using as dependent variable $\tilde{r}_t = \tilde{p}_t - ln(XR_t)$. However, these predictions can be improved.

Using this as a set of starting predictions, RRD, as shown in Rao et al. [2010a], embeds a re-written version of equation (13) in a re-writing (4) as follows,

$$y_t{=}Z_t p_t{+}B_t X_t \theta{+}\zeta_t \tag{14}$$

where, θ, a function of β_{0t} and β_s and B_t a mapping matrix to non-participating countries. Upon convergence of the estimation algorithm (which involves the Kalman filter algorithm) $\theta \to 0$ and (14) reduces to (4), which is then used by the Kalman filter and Smoothing algorithm to produce estimates of the latent vector p_t and an associated mean squared prediction error matrix. The point to note here is that unlike the PWT and other extrapolation methods, this approach generates predictions for all the cells (time periods and countries). However, it is trivial to limit the regression based PPPs, $\hat{p}_t$, (through Z_t and B_t) to be used by the model's predictor to only those countries and years when no ICP benchmark observations, $\tilde{p}_t$, are available.

The identification of p_t from the above mapping requires information on how PPPs evolve over time. The updating of PPPs from period $t-1$ to t is through the GDP deflators in the country concerned and in the reference country. Thus,

$$PPP_{i,t} = PPP_{i,t-1} \times \frac{GDPDef_{i,[t-1,t]}}{GDPDef_{1,[t-1,t]}} \tag{15}$$

Taking logarithms on both sides of (15), and assuming the updating equation (15) holds on average due to measurement error, we have

$$p_{it} = p_{i,t-1} + c_{it} + \eta_{it} \tag{16}$$

where $c_{it} = ln(\frac{GDPDef_{i,[t-1,t]}}{GDPDef_{US,[t-1,t]}})$; and η_{it} is random error accounting for measurement error in the growth rates. Equation (16) is commonly used in constructing panels of PPPs including the PWT and in the construction of the Maddison series[11]. The variance covariance matrix of η_{it} is assumed to be similar to the matrix in equation (8).

As the current problem is one of finding predictions for the vectors of PPPs from a variety of sources of noisy information through the ICP benchmarks; regression predictions and, finally, the updating equation in (16), a state-space (SS) representation is suitable for these kinds of problems and the approach proposed formulates all the information in equations (4) to (16) in the form of a set of observation and transition equations on the state vector p_t which is the vector of unknown $ln(PPP_t)$. Under Gaussian assumptions, the Kalman filter and Smoother predictor of the conditional mean, p^*_{it}, conditional on information available at time t, is a minimum squared error predictor of the state vector, p_t[12]. The panel of PPPs is the obtained by,

$$P\hat{P}P_{it} = \exp(p^*_{it})\; i = 1, ..., N \text{ and } t = 1, ..., T \tag{17}$$

3.2 Analytical properties of the Model

In order to provide a better appreciation of the features of the econometric model used here, a number of analytical results pertaining to the model are presented here but without proofs. In particular, these properties demonstrate the flexibility of the model and show how the model provides intuitively meaningful predictions under specific scenarios. The following properties are stated without proofs but complete proofs are provided in Rao et al. [2010a,b].

3.2.1 Constraining the model to track PPPs for countries participating in the benchmarks

As the ICP is the main source of PPPs for countries participating in different benchmarks and given that respective PPPs are determined using price data collected from extensive price surveys, one may consider it necessary that the econometric method proposed should generate predicted PPPs that are identical to PPPs for the countries participating in different ICP benchmarks. In the model proposed here, this can be achieved by simply setting the variance of the disturbance term in (8) to be equal to zero. In this case a particular property of Kalman filter predictions is that the predicted PPPs ($P\hat{P}P_{it}$) will be identical to the benchmark $P\tilde{P}P_{it}$[13] when t is a benchmark year.

[11] Maddison [2007] presents series that are extrapolated from the 1990 benchmark year.

[12] Technical details and equations for the Kalman Filter and Smoother are provided in Appendix A.6 and Appendix B of Rao et al. [2010a].

[13] This result follows from the work of Doran [1992].

3.2.2 Constraining the model to preserve movements in the Implicit GDP Deflator

In the currently available PWT and the Maddison series, growth rates in real GDP and movements in the implicit price deflators are preserved. As the GDP deflator data are provided by the countries and given that such deflators are compiled using extensive country-specific data, it is often considered more important that the predicted PPPs preserve the observed growth rates implicit in the GDP deflator. This essential feature can be guaranteed in RRD by simply stipulating the variance of the error in the updating equation (16) be zero. It is trivial to show that the national level movements in prices are preserved using the formulae for the fixed interval Kalman Smoother[14].

We note here that it is not possible to simultaneously constrain the predictors to track the benchmark PPPs as well as the national movements in GDP deflators. One has to choose either one or none of these restrictions when generating panels of extrapolated PPPs. Our recommended approach is to simply use unconstrained equations of our model and thereby not impose either of the restrictions described above.

3.2.3 Kalman Filter predictions as "weighted averages" of benchmark year only extrapolations

Proof of this important property is presented in Rao et al. [2010b]. Suppose there are $M + 1$ benchmark years. If regression based predictions are used to extrapolate PPPs to non-participating countries only in benchmark years and then use the implicit price deflators to extrapolate from one year to the next, then it is possible to construct a panel of extrapolated PPPs for each of the benchmark years. In this case, an obviously intuitive approach is to make use of an average of these $M + 1$ panels of PPPs. An important property of the RRD approach is that the predictions p_t^* can be shown to be a weighted average of the $M+1$ panels of PPPs, where the weights are determined by the diagonal elements of the 'Kalman Gain' matrices, which represent the gain in information provided by an additional benchmark. The weights can be interpreted as reflecting the reliability of the $j - th$ benchmark.

3.2.4 Invariance of the Predicted PPPs to the Choice of the Reference Country

The relative purchasing powers of currencies of countries should, in principle, be invariant to the choice of the reference country. It can be shown that RRD satisfy this important invariance property. The proof of this property is quite involved and it is presented in Appendix A of Rao et al. [2010a].

3.3 Empirical Results

In our recent update of results we cover 180 countries and the years 1970 to 2010. Detailed descriptions of the data used can be found in Rao et al. [2010b]. The PPP data used in the estimation of the regression model is an unbalanced panel formed by the ICP benchmarks 1970, 1973, 1975, 1980, 1980, 1985, 1990, 1993, 1996, 1999, 2002

[14] The proof of this property is provided in Appendix B of Rao et al. [2010a].

and 2005. Several features of the PPP data are noteworthy. The 1975 benchmark covered 34 countries. The 1980, 1985 and the recent 2005 benchmarks represent a truly global comparisons with PPPs computed using data for all the participating countries. The benchmark 1996 was a global comparison but it is generally considered weaker and less reliable than the earlier benchmarks as well as the most recent 2005 benchmark. The intervening benchmarks from 1990 cover only the OECD and EU countries. All the benchmarks prior to 1990 made use of the Geary-Khamis method of aggregation but since then the GiniEKS (GEKS) system has been the main aggregation procedure. In the current empirical analysis no adjustments have been made to the PPPs to account for the differences in the underlying aggregation methods.

The variables used in the regression model for national price levels (11) can be classified under two categories. As we employ a panel data regression model, a number of dummy variables designed to capture country-specific episodes that may influence the exchange rates or PPPs or both and to capture fixed effects are used. The second set of variables used are structural and drawn from the works of Kravis and Lipsey [1983], Clague [1988], Ahmad [1996], Bergstrand [1996] and Heston et al. [2006]. A complete list is available in from the authors[15].

The spatial autocorrelation, which is a special feature of our approach, is introduced through the spatial-weights matrix. The spatial weights are computed using bilateral trade flows (a number of alternative measures of economic distance were studied and details are presented in Rambaldi et al. [2010]). We find the estimate of the spatial correlation parameter to be 0.7 and statistically significant indicating the presence of strong positive spatial autocorrelation in the price level regression model.

3.3.1 UQICD website for PPPs and Real Incomes

The first set of predicted panel of PPPs from our project covered 141 countries and a 35-year period from 1971 to 2005. These PPPs along with standard errors are available through a dedicated website. As a part of the requirements of funding from the Australian Research Council, the website UQICD, the University of Queensland International Comparison Data, was established late in 2010 and was made publicly available in April, 2011. The URL for the website is: uqicd.economics.uq.edu.au. The website provides interactive tools to allow the user to choose the countries, years and variables to be downloaded.

Consistent with the general econometric approach described above, two sets of extrapolated PPPs are available. The first, which is our preferred option, provides extrapolations without imposing any prior restrictions with respect to tracking either the benchmarks or the implicit GDP deflators. The second series, however, is a PPP series which is constrained to preserve the movements in the implicit GDP deflator (see Section 3.2.2 for details).

A special feature of the website is the availability of comparative data in the form of easily interpretable charts. The new version of the predicted panel which we label "UQICD Mark II" covers 180 countries for the period 1970 -

[15] The new list differs significantly from the set of variables used in our earlier work, Rao et al. [2010b]

2010. Figure 1 presents estimates for four selected countries, viz., Australia, Brazil, China and India which include both UQICD Mark I (currently available through the website) and UQICD Mark II which will be released shortly[16]. The graphs also include the PWT 6.3 and PWT7.1 to provide a comparison[17].

Once the desired PPP series is chosen from the alternatives available, it is possible to compile real GDP aggregates at current prices by converting aggregates in national currency units into a common currency unit using equation (2), where PPP_{it} is replaced by $P\hat{P}P_{it}$ (see (17).

4 A Constrained GEKS Approach to Consistent Panels of PPPs Over Time and Space

In this section we pursue the GEKS approach to compile PPPs with time-space dimensions. We assume that PPP matrices (for cross-country comparisons at current prices) are available for each of the periods. In this case we assume that a procedure similar to RRD is already implemented and thus a panel of PPP^{*}_{it} $i = 1, \ldots, N$ and $t = 1, \ldots, T$ is available to start the proposed procedure.

In view of the space-time nature of the approach, we introduce further notation to what has been introduced in the previous sections. Let PPP^{ts}_{ik} denote the PPP for *country k in period s* expressed relative to the *reference country i and reference period t*. For example if $i = US$ and $t = 2000$, then all the PPP's are expressed in US 2000 dollars.

Let Π represent a $(TN \times TN)$ matrix of PPPs over space and time. Then we can write Π as

$$\Pi = \begin{bmatrix} \Pi^{11} & \Pi^{12} & \ldots & \Pi^{1T} \\ \Pi^{21} & \Pi^{22} & \ldots & \Pi^{2T} \\ \ldots & \ldots & \ldots & \ldots \\ \Pi^{T1} & \Pi^{T2} & \ldots & \Pi^{TT} \end{bmatrix} \tag{18}$$

where Π^{ts} represents an $N \times N$ matrix showing PPPs for countries in period s relative to countries in period t used as reference countries. PPP^{ts}_{ik} is a typical element of Π.

4.1 Elements of block-diagonal matrices

The matrix in equation (18) involves two broad categories of PPPs, those referring to PPPs across countries at a given point of time and other related to PPPs for countries over time. The first set refers to the block-diagonal

[16]We expect to release UQICD Mark II by the end of June, 2013

[17]We note that PWT 6.3 was generated without using the 2005 benchmark For a more appropriate comparison, it is necessary to compare the series generated using our approach but without using the 2005 benchmark with the series in the PWT (available through the already mentioned papers).

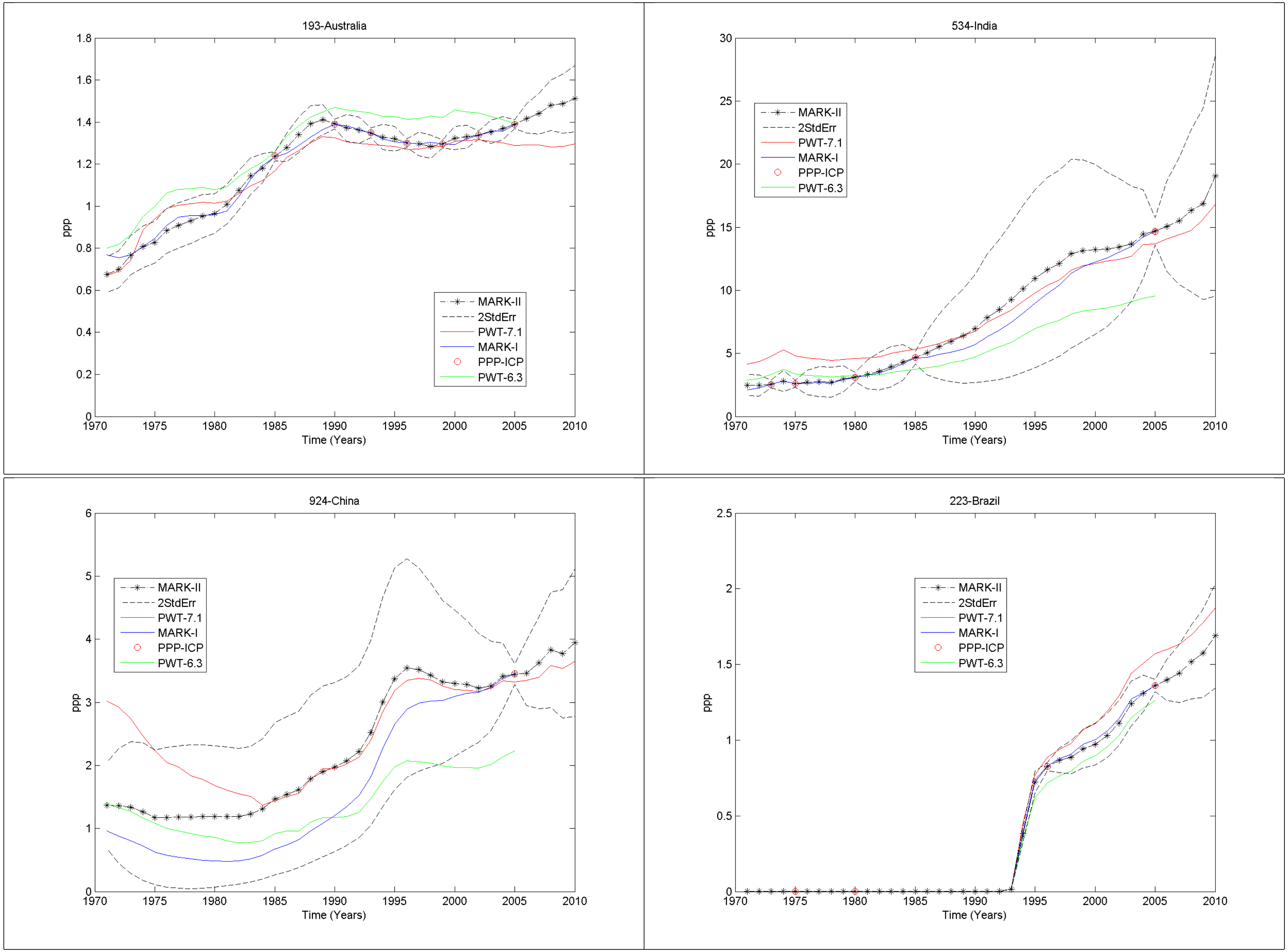

Figure 1: PPP at current prices from UQICD Mark I and Mark II (preliminary). Selected Countries

matrices Π^{tt}, for $t = 1, \ldots, T$. For example if t is the year 2005, then this matrix provides PPPs for all pairs of countries in benchmark year 2005. Thus PPPs in these block diagonal matrices represent *PPPs at current prices* (see Section 2 for a description of these concepts). We assume that the block diagonal matrices satisfy transitivity property. Transitivity of Π^{tt} implies the existence of a vector of constants, say $\pi^t = [\pi_1^t, \ldots, \pi_N^t]$ such that

$$PPP_{ik}^{tt} = \frac{\pi_k^t}{\pi_i^t} \; \forall i \text{ and } k \tag{19}$$

The principal source of information for these block diagonal matrices are studies like RRD or the PWT which, based on the ICP data, provide extrapolated PPPs that cover both benchmark and non-benchmark years and countries. Without loss of generality we can assume that matrices Π^{tt} satisfy transitivity for all t and, therefore, can be expressed in a form similar to (19).

4.2 Elements of off-diagonal matrices

Elements of the off-diagonal matrices are not directly observed nor are available from any of the standard extrapolation studies. Typical elements of these matrices refer to PPP for a country k in period s relative to a reference country i in the reference period t. We use the following procedure to fill the elements of the off-diagonal matrices Π^{ts}. Let us for example consider PPP_{ik}^{12} which is a typical element of Π^{12}. This is PPP for country k in period 2 relative to country i in period 1. We can derive this comparison either using a comparison between i and k in period 1 or in period 2. We can update the period 1 comparison, PPP_{ik}^{11}, using the implicit deflator d_k^{12} which represents movements in prices of country k from period 1 to 2. In this case, we have

$$PPP_{ik}^{12} = PPP_{ik}^{11} \times d_k^{12} \tag{20}$$

Alternatively, we could start with comparisons in period 2 and adjust PPP_{ik}^{22} backwards using d_i^{21} representing the implicit price deflator in country i measuring change from period 2 to period 1[18]. This in turn gives and alternative to (20) in the form:

$$PPP_{ik}^{12} = {}^{PPP_{ik}^{22}}\!/_{d_i^{21}} \tag{21}$$

if the deflators satisfy the time reversal test. As these two alternatives are equally satisfactory, we make use of the geometric mean of (20) and (21) to measure PPP_{ik}^{12}.

[18]Typically d_i^{21} is assumed to satisfy the time reversal test which is automatically satisfied if country i uses chained Fisher index in which case $d_i^{21} = {}^{1}\!/_{d_i^{12}}$.

$$PPP_{ik}^{12} = [(PPP_{ik}^{11} \times d_k^{12}) \times \left({}^{PPP_{ik}^{22}}/{}_{d_i^{21}}\right)]^{1/2} \tag{22}$$

Substituting (19) into (22), we can express the general element in the off-diagonal matrices in logarithmic form as:

$$\ln PPP_{ik}^{ts} = \frac{1}{2}\left[(\ln \pi_k^s - \ln \pi_i^s) + (\ln \pi_k^t - \ln \pi_i^t) + (\ln d_k^{ts} - \ln d_i^{st})\right] \tag{23}$$

Using the form in (22) we can fill all the off-diagonal blocks thus completing the matrix Π. We note that Π is not transitive. We propose to use the standard GEKS approach to construct transitive time-space comparisons and we introduce an additional constraint of spatial fixity in each of the years.

4.3 GEKS Methodology and transitivity

The GEKS methodology[19] involves the minimisation of sum of squared logarithmic differences between observed PPPs and the transitive PPPs solved out of the system. Let Π^* be a solution of the GEKS method. Then the typical elements of Π^*, PPP_{ik}^{*ts} are obtained under the GEKS procedure,

$$\min_{PPP^*} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{k=1}^{N}[\ln PPP_{ik}^{ts} - \ln PPP_{ik}^{*ts}]^2 \tag{24}$$

Subjecting to the transitivity of PPP_{ik}^{*ts}. In implementing GEKS and solving (24) we reparameterise the objective function by noting that the matrix Π^* is transitive if and only if there exists a vector π^* of order $(TN \times 1)$ with a typical element π_k^{*s}, associated country k and period s such that

$$\Pi_{ik}^{*ts} = \frac{\pi_k^{*s}}{\pi_i^{*t}} \ \forall i, k \text{ and } t, s \tag{25}$$

Substituting (25) into (24) yields the GEKS objective function in terms of the new parameters:

$$\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{k=1}^{N}\left[\ln PPP_{ik}^{ts} - \ln \pi_k^{*s} + \ln \pi_i^{*t}\right]^2 \tag{26}$$

Minimisation of (26) yields the standard GEKS solution to the problem. In the process we get a transitive matrix of PPPs which are time-space consistent. We note that in achieving time-space consistency, GEKS perturbs all the basic elements in the block-diagonal matrices in (18). In this paper we improve this process further by imposing additional restrictions on the solutions to ensure consistency of the time-space PPPs from (26) and the observed PPPs for each of the time periods, a form of *fixity*.

[19]See Diewert [2013] or Rao [2009] for a description of GEKS and its properties

4.4 GEKS with fixity condition

The main problem with a straightforward minimisation of (26) is that comparisons between countries at a given period of time obtained from GEKS will not be equal to the PPP's in the block diagonal of matrix in equation (18). Suppose we have international price comparisons in the form of PPPs for pairs of countries for a given year, say 2005. These comparisons are essentially price comparisons based on prices observed in 2005. When we minimise (26), the resulting comparisons between countries for the year 2005 will not be the same as those observed for 2005 as the new comparisons are affected by comparisons for all pairs of countries in all other periods in the exercise. This basically means that price and real income comparisons for 2005 will differ at current 2005 prices and constant 2005 prices which is not desirable. So we implement a refined GEKS by imposing the condition that the price (and hence real income) comparisons for a given year are the same at the current and constant prices. We note here that multiplying PPPs between countries in a given year by a constant does not alter price comparisons between countries which are in the form of ratios of PPPs. GEKS with fixity can be achieved by minimising (26) subject to additional restrictions:

$$\pi_k^{*s} = \delta^{*s} \pi_k^s \, \forall k \text{ and } s \tag{27}$$

where δ^{*s} $s = 1, ..., T$ is a set of constants.

If we incorporate restrictions (27), in log form, into (26), the GEKS method with $fixity$ requirement simplifies to one of

$$\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{k=1}^{N}\left[\ln PPP_{ik}^{ts} - \ln\delta^{*s} - \ln\pi_k^s + \ln\delta^{*t} + \ln\pi_i^t\right]^2$$

and the solution is given by

$$\min_{\{\delta^1,\delta^2,\ldots,\delta^T\}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i=1}^{N}\sum_{k=1}^{N}\left[\ln PPP_{ik}^{ts} - \delta^s - \ln\pi_k^s + \delta^t + \ln\pi_i^t\right]^2 \tag{28}$$

where $\delta^s = \ln\delta^{*s}$ $(s = 1, ..., T)$.

We note that if $\left\{\delta^{*1}, \delta^{*2}, ..., \delta^{*T}\right\}$ is a solution to the problem, $\left\{\kappa\delta^{*1}, \kappa\delta^{*2}, ..., \kappa\delta^{*T}\right\}$ for any κ >0 is also a solution to the problem. Hence we solve (28) after imposing an identifying restriction, $\delta^{*1} = 1$, which is the same as $\delta^1 = 0$. This means that all the comparisons are anchored on reference period 1. The final solution results in price and real income comparisons at *constant year 1 prices.*

The first order conditions for optimisation after imposing $\delta^1 = 0$ yield the following system of $(T - 1)$ linear equations:

$$\begin{bmatrix} 1 & -\frac{1}{T-1} & \cdots & -\frac{1}{T-1} \\ -\frac{1}{T-1} & 1 & \cdots & -\frac{1}{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{T-1} & -\frac{1}{T-1} & \cdots & 1 \end{bmatrix} \begin{bmatrix} \delta^2 \\ \delta^3 \\ \vdots \\ \delta^T \end{bmatrix} = \begin{bmatrix} \frac{1}{(T-1)N(N-1)} \sum_{s=1(\neq 2)}^{T} \sum_{i=1}^{N} \sum_{k=1}^{N} \left[\ln PPP_{ik}^{2s} - \ln \pi_k^s + \ln \pi_i^2\right] \\ \vdots \\ \vdots \\ \frac{1}{(T-1)N(N-1)} \sum_{s=1(\neq T)}^{T} \sum_{i=1}^{N} \sum_{k=1}^{N} \left[\ln PPP_{ik}^{Ts} - \ln \pi_k^s + \ln \pi_i^T\right] \end{bmatrix} \tag{29}$$

This leads to the following solution for the unknown constants.

$$\begin{bmatrix} \hat{\delta}^2 \\ \hat{\delta}^3 \\ \vdots \\ \hat{\delta}^T \end{bmatrix} = \left(\frac{T-1}{T}\right) \begin{bmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \cdots \\ 1 & 1 & \cdots & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{(T-1)N(N-1)} \sum_{s=1(\neq 2)}^{T} \sum_{i=1}^{N} \sum_{k=1}^{N} \left[\ln PPP_{ik}^{2s} - \ln \pi_k^s + \ln \pi_i^2\right] \\ \vdots \\ \vdots \\ \frac{1}{(T-1)N(N-1)} \sum_{s=1(\neq T)}^{T} \sum_{i=1}^{N} \sum_{k=1}^{N} \left[\ln PPP_{ik}^{Ts} - \ln \pi_k^s + \ln \pi_i^T\right] \end{bmatrix} \tag{30}$$

Now we can derive expressions for each of the elements of $\left\{\hat{\delta}^2, ..., \hat{\delta}^T\right\}$. We note further that typical elements involved in the summation on the RHS of equation (30) involve terms like:

$$\sum_{s=1(\neq 2)}^{T} \sum_{i=1}^{N} \sum_{k=1}^{N} \left[\ln PPP_{ik}^{2s} - \ln \pi_k^s + \ln \pi_i^2\right] \tag{31}$$

which can be further simplified by noting the procedure used in filling the elements of the off-diagonal blocks of matrices as described in equation (23). Inserting (23) in (31) allows us to derive simple closed form solutions for the elements of the vector $\left\{\hat{\delta}^2, ..., \hat{\delta}^T\right\}$. After tedious but fairly straightforward algebraic manipulations, we obtain the following simple expression for $\hat{\delta}^s$ $(s = 2, ..., T)$,

$$\hat{\delta}^s = \frac{1}{N} \sum_{i=1}^{N} \left[\frac{1}{T} \sum_{l=1}^{T} (\ln d_i^{1l} + \ln d_i^{ls})\right] \tag{32}$$

with $\hat{\delta}^1 = 0$

However, our main interest is in $\widehat{\delta^{*s}} = \exp\left(\hat{\delta}^s\right)$. Thus, we have

$$\widehat{\delta^{*1}} = 1 \tag{33}$$

$$\begin{aligned}\widehat{\delta^{*s}} &= \exp(\hat{\delta}^s) \\ &= \prod_{i=1}^{N}\left\{\prod_{l=1}^{T}\left[d_i^{1l} \times d_i^{ls}\right]^{1/T}\right\}^{1/N} \quad s = 2, ..., T\end{aligned}$$

The expression in (33) has a fairly simple interpretation. We can express it as follows,

$$\widehat{\delta^{*s}} = \prod_{i=1}^{N}\left\{GEKS_i^{1s}\right\}^{1/N} \tag{34}$$

where, $GEKS_i^{1s} = \prod_{l=1}^{T}\left[d_i^{1l} \times d_i^{ls}\right]^{1/T}$ is the GEKS comparison of price movements from period 1 (our reference period) to period s in country j.

Given this $\widehat{\delta^{*s}}$ is a simple unweighted geometric mean of country specific transitive measures of change in price from period 1 to $s(= 2, ..., T)$. These values can be used in constructing a time-space consistent matrix of PPPs over time and space with the property of fixity of cross-country comparisons within each period. We may note that adjustment factors $\widehat{\delta^{*s}}$ can be computed once we have data on domestic deflators, d_i^{ls} using the expression (33) which might be interpreted as a simple measure of global inflation over the period 1 to s.

4.5 Space-Time Consistent Panel of PPPs

Based on the solution for the equation in (33), the consistent panel of space-time PPPs constructed using the vector π^* with typical element π_i^{*t} for $i = 1, 2, ..., N$ and $t = 1, ..., T$ can be derived from equations (25) and (27):

$$PPP_{ik}^{*ts} = \frac{\pi_k^{*s}}{\pi_i^{*t}} = \frac{\exp(\ln \pi_k^s + \delta^s)}{\exp(\ln \pi_i^t + \delta^t)} = \frac{\exp(\ln \pi_k^s)}{\exp(\ln \pi_i^t)} \times \frac{\exp(\delta^s)}{\exp(\delta^t)}) \tag{35}$$

$$= = \frac{\pi_k^s \times \delta^{*s}}{\pi_i^t \times \delta^{*t}} \tag{36}$$

$$= \frac{PPP_k^{ss} \times \delta^{*s}}{PPP_i^{tt} \times \delta^{*t}} \tag{37}$$

The main property of this panel of PPPs is that it satisfies fixity of cross-country comparisons for each of the time periods. In Section 2 we defined real GDP expressed in constant prices ($CRGDP_{it}^{ks}$) as the GDP of country i in period t in base year s and using reference country k (see equation (3)). Using an estimate of (37) as the denominator in equation (3), i.e., to adjust the GDP of country i at time t. We note that the estimate of (37) would be obtained by replacing $PPP_i^{tt}(PPP_k^{ss})$ by $P\hat{P}P_{it}(P\hat{P}P_{ks})$ using (17) and $\delta^{*s}(\delta^{*t})$ by $\widehat{\delta^{*s}}(\widehat{\delta^{*t}})$ using (34).

5 Conclusions

The paper provides an overview of the status of the research on the development of an econometric approach to the construction of panels of PPPs for the purpose of spatio-temporal comparisons of prices and real incomes. In this paper we present the approach developed by our team to construct a panel of PPPs at current prices. We provide an overview of the method and present computed PPPs obtained using 180 countries and the period 1970-2010 from the soon to be released UQICD Mark II. A GEKS based method satisfying fixity of PPPs of currencies in a given year is proposed in the paper. We show the derived closed form solution which can be used to compute a full panel of space-time consistent PPPs.

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